Exercise 1: For a solid in equilibrium, verify if the following stress field is possible.

$$\sigma_{11} = -2C_1 x_1 x_2; \quad \sigma_{22} = C_2 x_3^2; \quad \sigma_{33} = C_2 x_3^2$$
$$\sigma_{23} = 0; \quad \sigma_{12} = C_1(C_2 - x_2^2) + C_3 x_1 x_3; \quad \sigma_{13} = -C_3 x_2$$

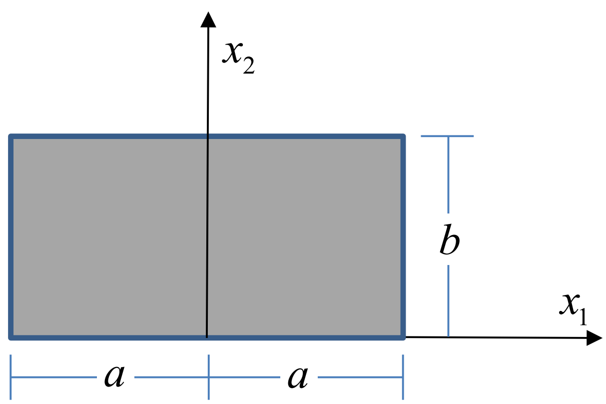
Solution:

We need to verify the equilibrium equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \quad \Rightarrow \quad -2C_1 x_2 - 2C_1 x_2 + 0 \neq 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0 \quad \Rightarrow \quad C_3 x_3 + 0 + 0 \neq 0$$
$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0 \quad \Rightarrow \quad 0 + 0 + 2C_2 x_3 \neq 0$$

They do not satisfy all three equilibrium eqs. Therefore the field is not possible.

Exercise 2: For a thin plate (thickness *t*) without body forces, a stress field is given by,



$$\sigma_{11} = p x_2 x_1^3 - 2c_1 x_2 x_1 + c_2 x_2$$
$$\sigma_{22} = p x_1 x_2^3 - 2 p x_1^3 x_2$$
$$\sigma_{12} = -\frac{3}{2} p x_1^2 x_2^2 + c_1 x_2^2 + \frac{1}{2} p x_1^4 + c_3$$

$p, c_1, c_2,$ and c_3 are constants.

(a) Examine if it represents a solution for a thin plate of thickness *t* shown in the figure.
(b) Find the resultant normal and shear forces on the boundaries $x_2 = 0,\ x_2 = b$ of the plate.

Solution:

We need to verify the equilibrium equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \quad \Rightarrow \quad p x_2 \left(3x_1^2\right) - 2c_1 x_2 - \frac{3}{2} p x_1^2 \left(2x_2\right) + 2c_1 x_2 = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \quad \Rightarrow -\frac{3}{2} p \left(2x_1\right) x_2^2 + \frac{1}{2} p (4x_1^3) + p x_1 (3x_2^2) - 2 p x_1^3 = 0$$

We see that the given field satisfies equilibrium.

To find the forces on the boundaries $x_2 = 0,\ x_2 = b$ we proceed as follows,

Edge $x_2 = 0$:

$$N_1 = \int_{-a}^{+a} \sigma_{12} t dx_1 = \int_{-a}^{+a} \left(\frac{1}{2} p x_1^4 + c_3 \right) t dx_1 = \frac{1}{5} p a^5 t + 2 c_3 a t$$

$$N_2 = \int_{-a}^{+a} \sigma_{22} t dx_1 = \int_{-a}^{+a} (0) t dx_1 = 0$$

Edge $x_2 = b$:

$$N_1 = \int_{-a}^{+a} \sigma_{12} t dx_1 = \int_{-a}^{+a} \left(-\frac{3}{2} p x_1^2 b + c_1 b + \frac{1}{2} p x_1^4 + c_3 \right) t dx_1$$

$$= -p a^3 b^2 t + \frac{1}{5} a^5 t + 2 a c_1 b^2 t + 2 a c_3 t$$

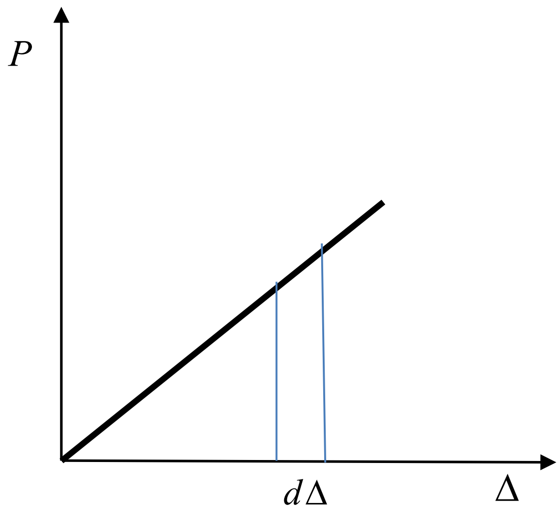
$$N_2 = \int_{-a}^{+a} \sigma_{22} t dx_1 = \int_{-a}^{+a} \left(p x_1 b^3 - 2 p x_1^3 b \right) t dx_1$$

$$= p b^3 t \left. \frac{x_1^2}{2} \right|_{-a}^{a} - 2 p b t \left. \frac{x_1^4}{4} \right|_{-a}^{+a} = 0$$

Exercise 3: A prismatic bar with a constant cross sectional area A is subjected to a uniform tensile stress as shown in the Figure below.

1. Find the strain energy U of the bar in terms of P, l, A, and E
2. Find the total elongation of the bar. What is the work done by the force P?
3. Find the total potential energy, defined as the difference of the strain energy and the work of the applied force, in terms of l, A, E and strain $\varepsilon = \sigma / A$.

Solution



$$U = \int_0^{\Delta} P d\Delta' \quad \text{with } P = \kappa\Delta$$

$$\Rightarrow U = \int_0^{\Delta} \kappa\Delta' d\Delta' = \frac{1}{2}\kappa\Delta^2 = \frac{1}{2}P\Delta$$

Hooks law for the bar,

$$\Delta = \frac{Pl}{EA} \Rightarrow U = \frac{1}{2}\frac{P^2 l}{EA}$$

Total elongation $\Delta = \varepsilon l = \frac{P}{A}\frac{l}{E}$

The work done by the force is $W = P\Delta$

The potential energy is defined as $\Pi = U - W = \frac{1}{2}P\Delta - P\Delta = -\frac{1}{2}P\Delta$

Exercise 4: A bar of uniform thickness, length L, area A and modulus E is loaded with two forces, P_1, P_2.

(1) Apply the forces slowly and simultaneously until their final values. Show that the strain energy in the bar is,

$$U(P_1, P_2) = \frac{1}{2}\frac{P_1^2 L}{AE} + \frac{1}{2}\frac{P_2^2 L}{AE} + \frac{P_1 P_2 L}{AE}$$

(2) Apply the forces in the following manner: Firstly, apply the force P_1 from zero to its final values and then the force P_2 from zero to its final value. Calculate the energy of the bar at the end of the experiment. Repeat the same experiment with P_2 applied first and

P_1 second. Show that the energy is the same and equal to that calculated in part (1).

(3) Show that the energy calculated in (1) or (2) is equal to the work done by the two applied forces, P_1, P_2

$$U(P_1, P_2) = \frac{1}{2} P_1 \Delta L_1 + \frac{1}{2} P_2 \Delta L_2$$

where ΔL_1, ΔL_2 are the total elongations due to the forces P_1, P_2 along their directions.

Solution:

1: Apply the two forces simultaneously.

The energy in the bar is

$$U(P_1, P_2) = \frac{(P_1 + P_1)^2 L}{2AE}$$ which is the given expression.

2: apply the forces one after the other

Apply $P_1 \rightarrow U_1 = \dfrac{P_1^2 L}{2AE}$

Apply $P_2 \rightarrow U_{12} = \dfrac{P_2^2 L}{2AE} + P_1 \dfrac{P_2 L}{AE}$

Summing up the two we obtain the total energy $U(P_1, P_2) = \dfrac{1}{2}\dfrac{P_1^2 L}{AE} + \dfrac{1}{2}\dfrac{P_2^2 L}{AE} + \dfrac{P_1 P_2 L}{AE}$.

Repeat with an inverse order,

Apply $P_2 \rightarrow U_2 = \dfrac{P_2^2 L}{2AE}$

Apply $P_1 \rightarrow U_{21} = \dfrac{P_1^2 L}{2AE} + P_2 \dfrac{P_1 L}{AE}$

Summing up the two we obtain $U(P_1, P_2) = \dfrac{1}{2}\dfrac{P_2^2 L}{AE} + \dfrac{1}{2}\dfrac{P_1^2 L}{AE} + \dfrac{P_2 P_1 L}{AE}$.

Note that the sequence of applying the forces does not change the final expression of energy.

3: From the second part of this Exercise we proceed as follows,

$$U(P_1, P_2) = \frac{1}{2} P_1 \frac{P_1 L}{AE} + \frac{1}{2} P_2 \frac{P_2 L}{AE} + \frac{1}{2} \frac{P_1 P_2 L}{AE} + \frac{1}{2} \frac{P_1 P_2 L}{AE}$$

$$= \frac{1}{2} P_1 (\Delta L)_{P_1} + \frac{1}{2} P_2 (\Delta L)_{P_2} + \frac{1}{2} P_2 (\Delta L)_{P_1} + \frac{1}{2} P_1 (\Delta L)_{P_2}$$

$$= \frac{1}{2} P_1 \left[(\Delta L)_{P_1} + (\Delta L)_{P_2} \right] + \frac{1}{2} P_2 \left[(\Delta L)_{P_2} + (\Delta L)_{P_1} \right]$$

$$= \frac{1}{2} P_1 (\Delta L)_1 + \frac{1}{2} P_2 (\Delta L)_2$$

This is an application of the theorem of work and energy (see Appendix A).

Problem 5: The strain energy of an elastic system, subjected to $P_1, P_2, ...P_n$ in equilibrium is given by the theorem of Clapeyron,

$$U = \frac{1}{2} a_{ij} P_i P_j \quad (i,j = 1,2,...n) \tag{a}$$

where the constants $a_{ij} = a_{ji}$ are the influence coefficients that relate the deflection at point *k* along the force P_k due to the entire system of applied forces *n*,

$$\delta_k = a_{kq} P_q. \tag{b}$$

Demonstrate that the derivative of (a) with reject to a force, gives the displacement along that force, i.e.,

$$\delta_{\boldsymbol{k}} = \frac{\partial U}{\partial P_k} \tag{c}$$

Solution:

We take the derivative of the energy (a) with respect to the force P_k along which we seek the deflection δ_k,

$$\begin{aligned}\frac{\partial U}{\partial P_k} &= \frac{1}{2} a_{ij} \frac{\partial P_i}{\partial P_k} P_j + \frac{1}{2} a_{ij} P_i \frac{\partial P_j}{\partial P_k} \\ &= \frac{1}{2} a_{ij} \delta_{ik} P_j + \frac{1}{2} a_{ij} P_i \delta_{jk} \\ &= \frac{1}{2} a_{kj} P_j + \frac{1}{2} a_{ik} P_i.\end{aligned}$$

Due to the symmetry of the influence coefficients and that *j* and *i* are dummy indices, the last expression becomes,

$$\frac{\partial U}{\partial P_k} = \frac{1}{2} a_{kj} P_j + \frac{1}{2} a_{ik} P_i = \frac{1}{2} a_{kj} P_j + \frac{1}{2} a_{ki} P_i = \frac{1}{2} a_{kj} P_j + \frac{1}{2} a_{kj} P_j = a_{kj} P_j. \tag{c}$$

Due to (b) the last result is the deflection,

$$\delta_k = \ \ a_{ki} P_i. \tag{d}$$

Problem 6: Show that for a solid of volume ω in equilibrium, the following relation holds,

$$\int_\omega \sigma_{ij} u_{i,j} dv = \int_\omega \sigma_{ij} \varepsilon_{ij} dv$$

where $\sigma_{ij}, \varepsilon_{ij}$ are the stress and strain components in the solid.

Solution: Due to the symmetry of the stress tensor

$$\sigma_{ij} = \sigma_{ji}$$

the strain-displacement relation,

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

and that the product of the two parameters is a scalar (i.e. we can swap the indices),

the integrant on the left-hand-side is expressed as

$$\sigma_{ij}u_{i,j} = \frac{1}{2}\left(\sigma_{ij}u_{i,j} + \sigma_{ij}u_{i,j}\right) = \frac{1}{2}\left(\sigma_{ij}u_{i,j} + \sigma_{ji}u_{j,i}\right) = \frac{1}{2}\left(\sigma_{ij}u_{i,j} + \sigma_{ij}u_{j,i}\right)$$

$$\Rightarrow \sigma_{ij}u_{i,j} = \frac{1}{2}\left(\sigma_{ij}u_{i,j} + \sigma_{ij}u_{j,i}\right) = \frac{1}{2}\sigma_{ij}\left(u_{i,j} + u_{j,i}\right) = \sigma_{ij}\varepsilon_{ij}$$

which is the integrant of the integral on the right-hand-side.